OSE.cpl: solution of the Orr-Sommerfield Equation via spectral collocation

Flavio Giannetti
fgiannetti@unisa.it

April 30, 2021

Abstract

Program OSE evaluates the eigenvalues of the Orr-Sommerfeld equation with a spectral collocation method using a 3-term recurrence relation to evaluate the derivatives of Chebyshev polynomials and the Fortran interface to call the appropriate lapack subroutine.

1 Formulation

The Orr-Sommerfeld equation is an eigenvalue equation describing the linear two-dimensional modes of disturbance to a viscous parallel flow. The solution to the Navier-Stokes equations for a parallel, laminar flow can become unstable if certain conditions on the flow are satisfied, and the Orr-Sommerfeld equation determines precisely what the conditions for hydrodynamic stability are. More details can be found in the book by Schmid and Henningson[1]

The configuration under investigation here is the classical Poiseuille flow in a plane channel characterised by a simple parabolic profile. While the base state is parallel, the perturbation velocity has components in both directions. The equation governing the evolution of small perturbations is derived assuming a total velocity of the form $u = (U_b(z) + u(x, z, t), 0, w(x, z, t))$ where $(U_b(z), 0, 0)$ is the parabolic velocity profile and linearising the Navier–Stokes equation around it. The perturbation velocity is assumed to have a wave-like solution $u \propto \exp(i\alpha(x - ct))$. Introducing such ansatz and the perturbation streamfunction ψ the following adimensional form of the Orr–Sommerfeld equation is obtained:

$$\frac{1}{i\alpha Re}(\frac{d^2}{dz^2} - \alpha^2)^2\psi = (U_b - c)(\frac{d^2}{dz^2} - \alpha^2)\psi - U_b''\psi \tag{1}$$

where

[1] *Stability and Transition in Shear Flows.* P.J. Schmid & D. S. Henningson. Springer-Verlag New York, Inc. , 2001

Re is the Reynolds number of the base flow and α is the wavenumber of the perturbation, which in the temporal stability problem is assumed to be real. The relevant boundary conditions for the perturbation are the usual no-slip conditions at the channel top and bottom wall $z = 1$ and $z = -1$ which can be expressed as

$$\alpha\psi = \frac{d\psi}{dz} = 0 \qquad \text{at both z=-1 and z=1} \tag{2}$$

The eigenvalue of the problem is c, while the corresponding eigenvector is ψ.

2 Numerical Solution

Solution of the OS equation can be obtained using different type of discretisation. Here we adopt a spectral collocation technique: the solution is sought in the form

$$\psi(z) = \sum_{0}^{N} a_n T_n(z) \tag{3}$$

where $T_n(z)$ indicates the Chebyshev polynomials of order N and a_n are $N + 1$ unknown coefficient to be determined. The derivatives of the Chebyshev polynomials are evaluated with a 3-term recurrence relation (Note: relation A44 in Schimd & Henningson has a typo. The correct formula is in line 50 of the cpl code). Other details concerning the discretisation can be found in the book. Spectral collocation forces the residual of the discretised equation to vanish at selected points, which are here taken to be the extrema (Gauss-Lobatto points) of the Chebyshev polynomials of order N-2. Forcing the residual to be zero at the interior points and imposition the boundary conditions 2 leads to a system of N+1 homogenous equation in N+1 unknowns a_n which admits non trivial solutions only for particular values of c. Once discretised the equations take the form of a generalised eigenvalue problem $A - cB$ which is then solved with the Lapack subroutine zggev. This is done by using the cpl instruction FORTRANCALL which translates the cpl into the fortran convention for parameters Figure 1 shows a typical result for the spectra obtained with the program OSE.cpl at $Re = 5772$ for $\alpha = 1.02$

By selecting with the mouse an eigenvalue on the graph, a new window is opened, showing the shape of the corresponding eigenvector (more precisely the first derivative of the eigenfunction which corresponds to the horizontal velocity. As an example, figure 2 shows the shape of the eigenvector corresponding to the leading eigenvalue

You can exit the plotting by pressing the right button of the mouse. Finally, in output the program prints the residual of the system of equations for each eigenvalues/eigenvectors.

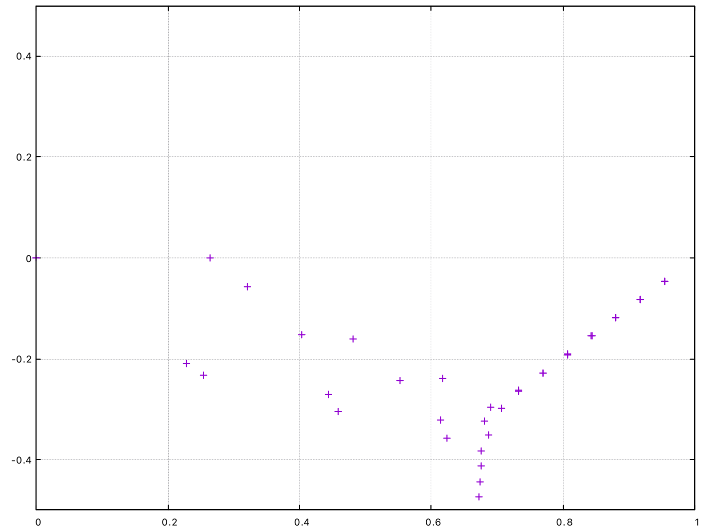

Figure 1: Spectra of the OSE equations at $Re = 5772$ and $\alpha = 1.02$ computed with the code OSE.cpl and N=50 polynomials

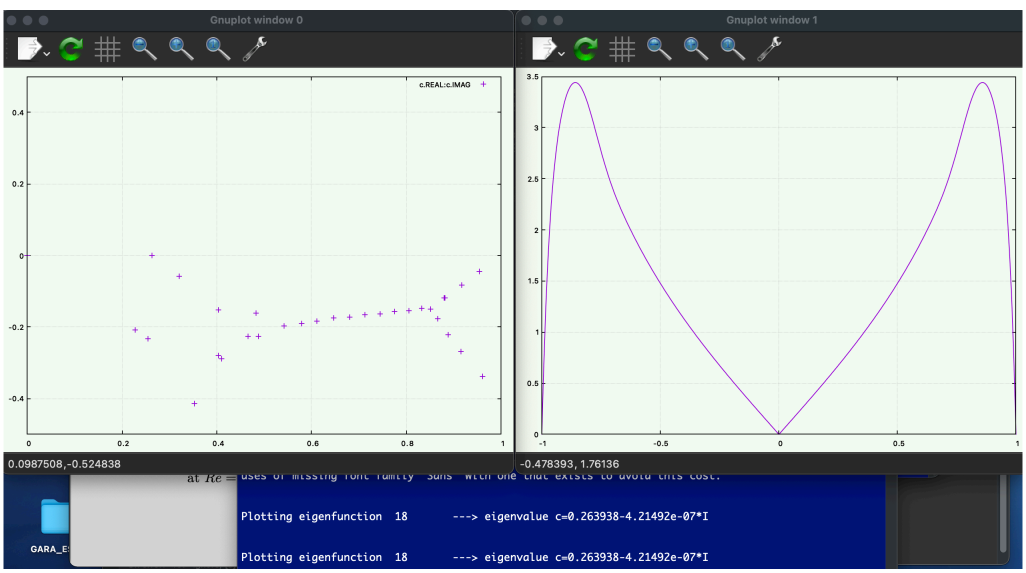

Figure 2: Shape of the first derivative of the eigenvector associated to the leading eigenvalue at $Re = 5772$ and $\alpha = 1.02$